Lecture 7 on Sept. 30

Today we firstly studied some general theory about rational functions. Then we considered a special rational function, linear transformation, which is the rational function of order 1

Example 1: . Given

$$R(z) = \frac{z^4}{z^3 - 1},$$

write it into the sum of partial fractions.

Solution: Step 1. by long division, we know that

$$R(z) = z + \frac{z}{z^3 - 1}.$$

Therefore we denote by $G(z) = z$ the polynomial part of $R(z)$ and $H(z) = z/(z^3 - 1)$ the proper rational part of $R(z)$;

Step 2. Supposing that ω_0 is a root of the polynomial $z^3 - 1$, we calculate that

$$H(\omega_0 + \frac{1}{\zeta}) = \frac{\omega_0\zeta^3 + \zeta^2}{3\omega_0^2\zeta^2 + 3\omega_0\zeta + 1} = \frac{1}{3\omega_0}\zeta + \text{proper rational function.}$$

The above calculations show that if ω_j is the j-th root of $z^3 - 1$, then $G_j(\zeta) = \zeta/(3\omega_j)$.

Step 3. By the above arguments, we can write $R(z)$ into the sum of partial fractions as follows:

$$R(z) = z + \sum_j \frac{1}{3\omega_j} \frac{1}{z - \omega_j}. \tag{0.1}$$

Now let us take a close look at (0.1). Usually in the theory of single variable functions, the domain of a rational function contains the points at where the denominator polynomial is non-zero. but from (0.1), when $z \to \omega_j$, z and $1/(z - \omega_i)$ converge to finite numbers for $i \neq j$. The only divergent term is $1/(z - \omega_j)$. Therefore it shows that when $z \to \omega_j$, $R(z) \to \infty$. So if ∞ is included in the range of $R(z)$, then we can allow ω_j lie in the domain of $R(z)$. This motivates us to extend the range of a rational function from $\mathbb{C}$ to the Riemann sphere $\mathbb{C} \bigcup \{\infty\}$. We can also extend the domain of $R(z)$ to the Riemann sphere. In fact, if $z = \infty$, the proper rational part of $R(z)$ in (0.1) equal 0. The only divergent term comes from the polynomial part of $R(z)$. That is z. So we know that as $z \to \infty$, $R(z) \to \infty$. Therefore we can define $R(\infty) = \infty$. More generally, we know that given a rational function, we can always regard it as a function from Riemann sphere to Riemann sphere. In fact if $R(z)$ is an arbitrary rational function, we can write it into the sum of partial fractions as follows

$$R(z) = G(z) + \sum_j G_j(\frac{1}{z - \beta_j}). \tag{0.2}$$

Here G and G_j are polynomials. for $z \neq \infty, \beta_j$, $R(z)$ is a finite number. If $z \to \beta_j$, then the term $G_j(1/(z - \beta_j))$ dominates. All the remaining terms approach to finite numbers. Moreove one can also show that $G_j(1/(z - \beta_j))$ approach to ∞ as $z \to \beta_j$. Then we can define $R(\beta_j) = \infty$. Samely we can define $R(\infty) = \infty$ if $G(z)$ is a non-constant polynomial.

Motivated by the above arguments, from now on, we always regard a rational function as a function defined on the Riemann sphere and taking its values in Riemann sphere. Moreover associated with a rational function $R(z)$, we define

Definition 0.1. *if p is a point such that $R(p) = \infty$, then we call p a pole point of $R(z)$. if q is a point such that $R(q) = 0$, then we call q a zero point of $R(z)$.*

In fact zeros and poles are quite related. if p is a zero of $R(z)$, then p must be a pole of the rational function $1/R(z)$. So in the following arguments, we focus on the pole points. Not just the definitions above, associated with any pole point, we can define a natural number by which the divergent rate of a rational function can be determined around its pole points.

Definition 0.2. *Noticing (0.2), when $z \to \beta_j$, the term $G_j(1/(z-\beta_j))$ dominates. So we define the order of β_j (denoted by $\mathrm{ord}(\beta_j)$) to be the order of the polynomial G_j. Samely we define the order of ∞ (denoted by $\mathrm{ord}(\infty)$), to be the order of the polynomial $G(z)$ if ∞ is a pole point of $R(z)$.*

Moreove we define

Definition 0.3. *Given a rational function $R(z)$, its order is defined by the summation of all orders of its pole points.*

Example 2. The order given in Definition 0.3 is consistent with the order of a polynomial if $R(z)$ is a polynomial.

Example 3. Using the rational function in Example 1, we see that it has four pole points $\omega_1, \omega_2, \omega_3, \infty$, where $\omega_1, \omega_2, \omega_3$ are the three roots of $z^3 - 1$. Since the polynomials $G(z)$ and G_j are all of order 1, then we know that $\mathrm{ord}(\infty) = \mathrm{ord}(\beta_j) = 1$. here $j = 1, ..., 3$. Therefore the order of the rational function is 4.

Example 4. The rational functions of order 0 are just constant functions.

Now we begin to study the rational functions of order 1. That is the so-called linear transformation. Noticing that if the order of a rational function is 1, then by the sum of partial fractions in (0.2) we know that $\mathrm{ord}(\infty) + \sum_j \mathrm{ord}(\beta_j) = 1$. Therefore only the following two cases may happen:

Case 1: there is no β_js and $\mathrm{ord}(\infty) = 1$;

Case 2: ∞ is not a pole and there is only one element in the set $\{\beta_j\}$ whose order is 1.

Obviously, the Case 1 corresponds to the linear function $az + b$ where a is a non-zero complex number. Rational functions in Case 2 share a general form

$$C_1 + \frac{C_2}{z - \beta},$$

where $C_2 \neq 0$. One can easily show that rational functions in Cases 1 and 2 can all be written as

$$\frac{az+b}{cz+d}, \qquad \text{with } ad \neq bc. \tag{0.3}$$

In the following, a rational function is called linear transformation if (0.3) holds. One of the most important properties of linear transformations is the theorem shown as follows

Theorem 0.4. *Linear transformation maps circles to circles.*

To show this theorem, we need a sort of preparations.

Proposition 0.5. *Linear transformation is invertible.*

Proof. The proof is just a straightforward calculation. Given $w = (az + b)/(cz + d)$, we can solve z by w as follows $z = (-dw + b)/(cw - a)$, provided that $w \neq a/c$. If $w = a/c$, then $z = \infty$. □

The second proposition is

Proposition 0.6. *Compsition of two linear transformations are also linear transformations.*

The proof is trivial. One can try the following example by yourself

Example 5. Let $T_1 = iz/(z+2)$, $T_2 = z/(z+1)$. Find out T_1T_2 and T_2T_1.

Proposition 0.7. *Given three distinct points in the Riemann sphere, denoted by z_2, z_3 and z_4, there is a unique linear transformation which maps (z_2, z_3, z_4) to $(1, 0, \infty)$*

Proof. Clearly

$$Sz = \frac{z - z_3}{z - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4} \tag{0.4}$$

is a linear transformation which maps (z_2, z_3, z_4) to $(1, 0, \infty)$. If S_1 and S_2 are two linear transformations which map (z_2, z_3, z_4) to $(1, 0, \infty)$, then by Propositions 0.5 and 0.6, $S_1S_2^{-1}$ is a linear transformation and moreover it maps $(1, 0, \infty)$ to $(1, 0, \infty)$. Assume

$$S_1S_2^{-1}(z) = \frac{az + b}{cz + d}.$$

then clearly $S_1S_2^{-1}(\infty) = a/c = \infty$. This fact shows that $c = 0$. Therefore we can assume $S_1S_2^{-1}(z) = a_1z + b_1$. When $z = 0$, it holds that $S_1S_2^{-1}(0) = b_1 = 0$. When $z = 1$, it holds that $S_1S_2^{-1}(1) = a_1 = 1$. All the above arguments show that $S_1S_2^{-1}(z) = z$ for all z a complex number. In other words, $S_1S_2^{-1}$ is an identity map. □

Definition 0.8. *We also define (z, z_2, z_3, z_4) to be the right-hand side of (0.4). Conventionally (z, z_2, z_3, z_4) is called cross-ratio of the four numbers z, z_2, z_3 and z_4. one should know that the value of the cross-ratio (z, z_2, z_3, z_4) is evaluated as follows: using z_2, z_3 and z_4, we can find a linear transformation by Proposition 0.7. We denote this linear transformation by S. The cross-ration is obtained by evaluating S at z.*

The cross-ration has two important properties.

Proposition 0.9. *For any linear transformation T, $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.*

Proof. Letting $Sz = (z, z_2, z_3, z_4)$, then one can show that ST^{-1} is a map which sends (Tz_2, Tz_3, Tz_4) to $(1, 0, \infty)$. Therefore by Proposition 0.7, we know that $ST^{-1}(w) = (w, Tz_2, Tz_3, Tz_4)$ for any complex number w. Setting $w = Tz_1$, the proof is finished. □

The second property associated with cross-ratio is

Proposition 0.10. $\text{Im}(z_1, z_2, z_3, z_4) = 0$ *if and only if the four points z_1, z_2, z_3 and z_4 lie on the same circle or straight line.*

Proof. we sketch the proof. If the four points lie on a same circle, then we know that the angle $\angle z_3z_2z_4$ equals to the angle $\angle z_3z_1z_4$. Clearly $\angle z_3z_2z_4$ is given by the argument of $(z_2 - z_3)/(z_2 - z_4)$. $\angle z_3z_1z_4$ is given by the argument of $(z_1 - z_3)/(z_1 - z_4)$. Therefore we know that $\arg((z_2 - z_3)/(z_2 - z_4)) = \arg((z_1 - z_3)/(z_1 - z_4))$. This equivalently shows that $\text{Im}(z_1, z_2, z_3, z_4) = 0$. □

We are now ready to prove Theorem 0.4.

Proof of Theorem 0.4. Fixing z_2, z_3, z_4 on a circle C, Tz_2, Tz_3, Tz_4 also determine a circle, say C'. Here T is a linear transformation. Choosing z an arbitrary point on C, then by Proposition 0.10, we have $\text{Im}(z, z_2, z_3, z_4) = 0$. Using Proposition 0.9, it also holds $\text{Im}(Tz, Tz_2, Tz_3, Tz_4) = \text{Im}(z, z_2, z_3.z_4) = 0$. Still by Proposition 0.10, Tz should lie on the circle C'. The proof is done. $\square$